DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ARITHMETIC.

46. Proposed by T. W. PALMER, Professor of Mathematics, University of Alabama.

A borrows \$500 from a building and loan association and agrees to pay \$9.50 per month for 72 months, the first payment to be made at the end of the first month. What rate of interest does he pay? The association claims to charge only 8% (the legal rate in Alabama). How can the per cent. be figured out?

Solution of unsolved problem in Vol. II, p. 74, by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

Let r=rate per month, $12r$=rate per annum, p=sum borrowed, n= number of payments, q=cash payment. Then, from Algebra, we get

$$q=\frac{pr(1+r)^n}{(1+r)^n-1},\quad q=9\tfrac{1}{2},\quad p=\$500,\quad n=72.$$

$\therefore (q-pr)(1+r)^n=q$, and $(19-1000r)(1+r)^{72}=19$.
$\therefore r=.00911$, and $12r=.10932=10.932\%$ per annum.

ALGEBRA.

293. Proposed by C. E. WHITE, Vanderbilt University, Nashville, Tenn.

Prove by mathematical induction that $\dfrac{(x-a)^{m-1}}{(m-1)!}f^{m-1}(a)+\dfrac{(x-a)^{m-2}}{(m-2)!}+\dots+\dfrac{(x-a)^2}{2!}f''(a)+(x-a)f'(a)+f(a)$ will be the remainder when $f(x)$ is divided by $(x-a)^m$.

Solution by the PROPOSER.

If $\phi(x)$ be the quotient found by dividing $f(x)$ by $(x-a)$ we can write the identity

$$\frac{f(x)}{x-a}=\phi(x)+\frac{f(a)}{x-a}.$$

Differentiating both members and solving for $f(x)/(x-a)^2$,

$$\frac{f(x)}{(x-a)^2}=\frac{f'(x)}{x-a}+\frac{f(a)}{(x-a)^2}-\phi'(x). \tag{1}$$

Since $\phi(x)$ is integral,

$$F\left[\frac{f(x)}{(x-a)^2}\right]=F\left[\frac{f(x)}{x-a}\right]+F\left[\frac{f(a)}{(x-a)^2}\right], \tag{2}$$

where $F[\ ]$ represents the fractional part of the quotient found by performing the division indicated within the brackets.

When the two fractions in the second member of (2) are added, the numerator of the resulting fraction equals the numerator of the fraction in the first member.

Hence,

$$R\left[\frac{f(x)}{(x-a)^2}\right]=(x-a)R\left[\frac{f'(x)}{x-a}\right]+R\left[\frac{f'(a)}{(x-a)^2}\right],$$

where $R[\ ]$ represents the fractional part of the remainder found by performing the division indicated within the brackets.

$$\therefore R\left[\frac{f(x)}{(x-a)^2}\right]=(x-a)f'(a)+f(a). \tag{a}$$

Differentiating (1) and solving for $\dfrac{f(x)}{(x-a)^3}$,

$$\frac{f(x)}{(x-a)^3}=-\frac{f''(x)}{2!(x-a)}+\frac{f'(x)}{(x-a)^2}+\frac{f(a)}{(x-a)^3}+\frac{\phi''(x)}{2!}. \tag{3}$$

$$\therefore F\left[\frac{f(x)}{(x-a)^2}\right]=-F\left[\frac{f''(x)}{2!(x-a)}\right]+F\left[\frac{f'(x)}{(x-a)^2}\right]+F\left[\frac{f(a)}{(x-a)^3}\right]$$

and $R\left[\dfrac{f(x)}{(x-a)^3}\right]=-\dfrac{(x-a)^2}{2!}R\left[\dfrac{f''(x)}{x-a}\right]+(x-a)R\left[\dfrac{f'(x)}{(x-a)^2}\right]+R\left[\dfrac{f(a)}{(x-a)^3}\right].$

Since $R\left[\dfrac{f''(x)}{x-a}\right]=f''(a)$, $R\left[\dfrac{f(a)}{(x-a)^3}\right]=f(a)$, and $R\left[\dfrac{f'(x)}{(x-a)^2}\right]$

$$=(x-a)f''(a)f'(a),$$

$$R\left[\frac{f(x)}{(x-a)^3}\right]=\frac{(x-a)^2}{2!}f''(a)+(x-a)f'(a)+f(a). \tag{b}$$

Equations (a) and (b) prove the theorem true when $m=2$ or 3. We will now assume it true for $m\leqq r$ and prove it true for $m=r+1$, or that

$$R\left[\frac{f(x)}{(x-a)^{r+1}}\right]=\frac{1}{r!}(x-a)^r f^r(a)+R\left[\frac{f(x)}{(x-a)^r}\right].$$

By mathematical induction,

$$\frac{f(x)}{(x-a)^m}=\frac{(-1)^m f^{m-1}(x)}{(m-1)!(x-a)}+\frac{(-1)^{m-1} f^{m-2}(x)}{(m-2)!(x-a)^2}+\dots$$

$$+\frac{f'(x)}{(x-a)^{m-1}}+\frac{f(a)}{(x-a)^m}+\frac{(-1)^{m-1}\phi^{m-1}(x)}{(m-1)!}. \qquad (4)$$

If we let $m=r$ and $m=r+1$ in (4), we derive

$$R\left[\frac{f(x)}{(x-a)^r}\right]=R\left[\frac{f(a)}{(x-a)^r}\right]+(x-a)R\left[\frac{f'(x)}{(x-a)^{r-1}}\right]-\frac{(x-a)^2}{2!}R\left[\frac{f''(x)}{(x-a)^{r-1}}\right]+$$

$$\dots(-1)^r\frac{(x-a)^{r-1}}{(r-1)!}R\left[\frac{f^{r-1}(x)}{x-a}\right], \qquad (5)$$

and $R\left[\frac{f(x)}{(x-a)^{r+1}}\right]=R\left[\frac{f(a)}{(x-a)^{r+1}}\right]+(x-a)\,R\left[\frac{f'(x)}{(x-a)^r}\right]$

$$-\frac{(x-a)^2}{2!}R\left[\frac{f''(x)}{(x-a)^{r-1}}\right]+\dots(-1)^{r+1}\frac{(x-a)^r}{r!}R\left[\frac{f^r(x)}{x-a}\right]. \qquad (6)$$

The second member of (6) can be derived from the second member of (5) by dividing the fraction in the bracket of each term by $(x-a)$ and adding the term $\frac{(-1)^{r+1}(x-a)^r}{r!}R\left[\frac{f^r(x)}{x-a}\right]$.

Corresponding to the term $(-1)^{s-1}\frac{(x-a)^s}{s!}R\left[\frac{f^s(x)}{(x-a)^{r-s}}\right]$ in (5), we have the term $(-1)^{s-1}\frac{(x-a)^s}{s!}R\left[\frac{f^s(x)}{(x-a)^{r-s+1}}\right]$ in (6).

Since the value of m in each bracket of the second member of (6) is less than $r+1$, and since for any value of $m\leq r$ the remainder can be found from the remainder for the preceding value of m by adding a term of degree one less than the value of m, therefore,

$$(-1)^{s-1}\frac{(x-a)^s}{s!}R\left[\frac{f^s(x)}{(x-a)^{r-s+1}}\right]=(-1)^{s-1}\frac{(x-a)^s}{s!}\left(\frac{(x-a)^{r-s}}{(r-s)!}f^{r-s+1}(a)\right.$$

$$\left.+R\left[\frac{f^s(x)}{(x-a)^{r-s}}\right]\right)=\frac{(-1)^{s-1}}{s!}\frac{(x-a)^r}{(r-s)!}f^r(a)+(-1)^{s-1}\frac{(x-a)^s}{s!}R\left[\frac{f^s(x)}{(x-a)^{r-s}}\right].$$

In (6), the term $\frac{(-1)^{r-1}(x-a)^r}{r!}R\left[\frac{f^r(x)}{x-a}\right]=(-1)^{r-1}\frac{(x-a)^r}{r!}f^r(a)$.

Hence, $R\left[\frac{f(x)}{(x-a)^{r+1}}\right]=(-1)^{r-1}\frac{(x-a)^r}{r!}f^r(a)+\sum_{s=1}^{s=r-1}(-1)^{s-1}\frac{(x-a)^r}{s!\,(r-s)!}f^r(a)$

$$+\sum_{s=1}^{s=r-1}(-1)^{s-1}\frac{(x-a)^s}{s!}R\left[\frac{f^s(x)}{(x-a)^{r-s}}\right]=(-1)^{r-1}\frac{(x-a)^r}{r!}f^r(a)$$

$$+\sum_{s=1}^{s=r-1}(-1)^{s-1}\frac{(x-a)^r}{s!\,(r-s)!}f^r(a)+R\left[\frac{f(x)}{(x-a)^r}\right].$$

It remains to be proved that

$$(-1)^{r-1}\frac{(x-a)^r}{r!}f^r(a)+\sum_{s=1}^{s=r-1}\frac{(-1)^{s-1}(x-a)^r}{s!\,(r-s)!}f^r(a)=\frac{(x-a)^r}{r!}f^r(a)$$

$$\text{or that } \frac{(-1)^{r-1}}{r!}+\sum_{s=1}^{s=r-1}\frac{(-1)^{s-1}}{s!(r-s)!}=\frac{1}{r!}. \tag{7}$$

The factorial series $\sum_{s=1}^{s=r-1}\frac{(-1)^{s-1}}{s!\,(r-s)!}=\frac{1}{(r-1)!}-\frac{1}{2!(r-2)!}+\frac{1}{3!(r-3)!}-$

$$\cdots\frac{(-1)^{r-2}}{(r-2)!2!}+\frac{(-1)^r}{(r-1)!}. \tag{8}$$

Multiplying (8) by $r!/r!$ and adding and subtracting $1/r!$, the series

$$=\frac{1}{r!}+\frac{1}{r!}[(r-1)-\frac{r(r-1)}{2!}+\frac{r(r-1)\,(r-2)}{3!}+\ldots\ (r-1)\ \text{terms}].$$

The sum of the terms in the bracket $=(-1)^r\frac{(r-1)!}{(r-1)!}=+1$ if r be even and -1 if r be odd.

Hence, (7) is true for all integral values of r.

Since we have proved the theorem true for $m=2$ and 3, it must be true for $m=4$, 5, 6, and for any integral value of m.

It will be observed that the limiting value of the remainder is Taylor's expansion of the function.

Also solved by H. V. Spunar and G. B. M. Zerr.